Journal of Heat Transfer

Technical Notes

This section contains shorter technical papers. These shorter papers will be subjected to the same review process as that for full papers.

Closed-Form Solutions for Transcendental Equations of Heat Transfer

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1 Introduction

In an era with modern computers and well-established root-finding algorithms, the numerical solution of transcendental equations is relatively straightforward. It may be of interest to heat transfer researchers and students, however, that it is also possible to state explicit solutions for transcendental equations arising in a variety of heat transfer problems. Explicit solutions to many transcendental equations have been developed for a variety of non-heat-transfer applications, such as celestial mechanics [the solution of Kepler's equation for elliptic and hyperbolic orbits (Siewert and Burniston, 1972)], ferromagnetism [the molecular field equation (Siewert and Essig, 1973)], nuclear reactor theory [the "critical condition" for a bare reactor (Siewert, 1973)], and applied mechanics [the eigenvalues of a clamped plate (Siewert and Phelps, 1978)]. The general theory for solving these problems is based on the methods of Muskhelishvili (1953) and has been developed by Burniston and Siewert (1973).

The procedure for solving transcendental equations for roots $\pm z_m$, $m = 0, 1, \ldots$, depends on formulating an appropriate Riemann problem of complex variable theory and then expressing the solution(s) of the transcendental equation in terms of a canonical solution of that problem. For the Riemann problem a function $\Omega_m(z)$ that is analytic in the complex plane except for the branch cut $[-1, 1]$ is separated into a product of functions,

$$\Omega_m(z) = \Lambda_m(z)\Lambda_m(-z), \tag{1}$$

(i.e., the so-called Wiener–Hopf factorization), where $\Lambda_m(z)$ is analytic in the complex plane except for the branch cut $[0, 1]$. The Riemann problem is defined by the boundary condition

$$\Phi_m^+(x) = \Theta_m(x)\Phi_m^-(x), \quad x \in (0, 1), \tag{2}$$

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where the superscripts + and − denote the approach of z to the real axis $x \in (0, 1)$ from above and below, and

$$\Theta_m(x) = \frac{\Omega_m^+(x)}{\Omega_m^-(x)} = \exp[2i \arg \Omega_m^+(x)]. \tag{3}$$

Here and elsewhere, $\arg \Omega_m^+(x) = \tan^{-1} [\mathrm{Im}\, \Omega_m^+(x)/\mathrm{Re}\Omega_m^+(x)]$. The objective is to find a function $\Phi_m(z)$ that is analytic in the plane cut from 0 to 1 along the real axis and nonvanishing in the finite plane so that $\Theta_m(x)$ will be continuous and nonvanishing for $x \in (0, 1)$. Then the canonical solution for $\Phi_m(z)$ can be written as (Burniston and Siewert, 1973)

$$\Phi_m(z) = (1 - z)^{-\kappa_m} \exp\left[\frac{1}{\pi}\int_0^1 \arg \Omega_m^+(x)\frac{dx}{x - z}\right], \tag{4}$$

with $\arg \Omega_m^+(0) = 0$. The index κ_m is such that $2\pi\kappa_m$ is the change in the argument of $\Theta_m(x)$ as x varies from 0 to 1.

Our objective here is to bring together in one place some existing closed-form solutions of transcendental equations, available in mathematics literature, and show they are applicable to heat transfer. In the process we illustrate that the transcendental equation for the convectively cooled thin fin problem is a special case of a transcendental equation arising in a radiative heat transfer problem. A secondary objective is to check the closed-form solutions numerically.

We illustrate in Sec. 2 the general procedure for solving such problems with the solution of the transcendental equation arising in the convective heat loss from a thin fin. In Sec. 3 we provide the closed-form solutions for two one-dimensional heat conduction problems in rectangular geometry. Then in Sec. 4 the solutions of the Wien displacement law and the asymptotic eigenvalue of the radiative transfer equation are given. Section 5 contains a few comments about the numerical evaluation of the closed-form equations.

2 Convectively Cooled Thin Fin Problem

The fin we consider is thin enough that there is only one dimension in which the temperature $T(x)$ varies along $0 \le x \le L$ due to a convective heat loss along the perimeter $\mathcal{P}$. The temperature at the base of the fin is a given value $T(x) = T_0$ and the environment is at an external temperature T_e. The fin cross-sectional area is A, its thermal conductivity is k, and the constant convective heat transfer coefficient is h_c. The governing partial differential equation is

$$\frac{d^2\theta(x)}{d^2x} - \beta^2\theta(x) = 0, \tag{5}$$

where $\theta(x) = [T(x) - T_e]/[T_0 - T_e]$ and where $\beta^2 = h_c\mathcal{P}/kA$.

We shall assume the fin has an insulated tip so that $dT(x)/dx|_{x=L} = 0$. The equation for the fin efficiency η, $0 \le \eta \le 1$, is given in introductory heat transfer textbooks as (Holman, 1990; Incropera and DeWitt, 1996; Mills, 1993)

$$\eta = \frac{\dot{Q}}{h_c \mathcal{P} L(T_0 - T_e)} = \frac{1}{\chi} \tanh(\chi), \tag{6}$$

for $\chi = \beta L$, and where the overall heat transfer from the fin is $\dot{Q}$. This is a transcendental equation if we seek L for a specified η. The equation can be rearranged so that we seek the zeros of the function

$$\begin{aligned}\Lambda(z) &= 1 - \eta z \tanh^{-1}(1/z),\\ &= 1 - \frac{\eta z}{2}\int_{-1}^{1} \frac{d\mu}{z - \mu},\end{aligned} \tag{7}$$

where $z = 1/\eta\chi$.

Because the roots of Eq. (7) are the two eigenvalues $\pm\nu_0$ of the radiative transfer equation for isotropic scattering (Case and Zweifel, 1967), the function $\Lambda(z)$ has been extensively investigated and the eigenvalues were tabulated many years ago (Case et al., 1953). We can obtain closed-form solutions for $\chi = (\eta\nu_0)^{-1}$ after taking over from Siewert (1980) the previously developed solutions for ν_0, which will now be briefly summarized.

$\Lambda(z)$ is analytic in the complex plane cut from -1 to 1 along the real axis, and satisfies the equations $\Lambda(0) = 1$, $\Lambda(\infty) = 1 - \eta$, and, for $|z| > 1$, $\Lambda(z) = \Lambda(\infty) - \eta(z^{-2}/3 + z^{-4}/5 + \ldots)$. The boundary values as the function approaches the cut from above/below are given by

$$\begin{aligned}\Lambda^{\pm}(\nu) &= \lim_{\epsilon \to 0^+} \Lambda(\nu \pm i\epsilon)\\ &= \lambda(\nu) \pm \pi i \eta \nu/2, \quad \nu \in (-1, 1),\end{aligned} \tag{8}$$

where

$$\lambda(\nu) = 1 - \eta\nu \tanh^{-1}\nu, \quad \nu \in (-1, 1). \tag{9}$$

Since $\Lambda(z)$ vanishes at $\pm\nu_0$, the function $\Lambda_0(z) = [\Lambda(\infty)]^{1/2} \times \Phi_0(z)(\nu_0 - z)$ is chosen to satisfy Eq. (1). Then the function $\Phi_0(z)$ is continuous and nonvanishing for $x \in (0, 1)$ and satisfies Eq. (2) with index $\kappa_1 = 1$ since 2π is the change in the argument of $\Theta_0(x)$ as x varies from 0 to 1. Thus from Eqs. (4) and (8),

$$\Phi_0(z) = (1 - z)^{-1} \exp\left[\frac{1}{\pi}\int_0^1 \tan^{-1}\left(\frac{\pi\eta\nu/2}{\lambda(\nu)}\right)\frac{d\nu}{\nu - z}\right]. \tag{10}$$

Because

$$\Omega(z) = \Lambda(\infty)\Phi_0(z)\Phi_0(-z)(\nu_0^2 - z^2), \tag{11}$$

it follows that the desired root we seek satisfies the equation

$$\nu_0^2 = z^2 + \Lambda(z)[\Lambda(\infty)\Phi_0(z)\Phi_0(-z)]^{-1}. \tag{12}$$

From this equation Siewert (1980) obtained two particularly concise equations for ν_0^2 by setting $z = 0$ and by letting $z \to \infty$, respectively:

$$\nu_0^2 = \frac{1}{\Lambda(\infty)} \exp\left[-\frac{2}{\pi}\int_0^1 \tan^{-1}\left(\frac{\pi\eta\nu/2}{\lambda(\nu)}\right)\frac{d\nu}{\nu}\right], \tag{13a}$$

$$\nu_0^2 = 1 + \frac{\eta}{3\Lambda(\infty)} - \frac{2}{\pi}\int_0^1 \tan^{-1}\left(\frac{\pi\eta\nu/2}{\lambda(\nu)}\right)\nu d\nu. \tag{13b}$$

From either of these equations the value of ν_0 can be computed, which can be used to determine $\chi = \beta L = (\eta\nu_0)^{-1}$.

3 Heat Conduction Problems

We turn now to the consideration of transcendental equations that arise in the solutions of one-dimensional heat conduction problems using the separation of variables technique. We consider the solutions of the problems (Özışık, 1989, 1993)

$$\frac{d^2T(x)}{dx^2} + \beta^2 T(x) = 0, \quad 0 < x < L, \tag{14}$$

$$-\alpha_1 \frac{dT(x)}{dx} + H_1 T(x) = 0, \quad \text{for} \quad x = 0, \tag{15}$$

$$\alpha_2 \frac{dT(x)}{dx} + H_2 T(x) = 0, \quad \text{for} \quad x = L, \tag{16}$$

where $\alpha_j = 0$ or 1 and $H_j = 0$, 1, or h_{cj}/k, where h_{cj} is the convective heat transfer coefficient at surface $j = 1$ or 2 and k_j is the corresponding thermal conductivity. We seek the eigenvalues β_m that are the roots of different transcendental equations depending on the values of the coefficients α_j and H_j.

One Insulated Surface. We consider the case where $\alpha_1 = \alpha_2 = 1$ and either $H_1 = H$ or $H_2 = H$. Then the equation for the eigenvalues $\pm\beta_m$ is (Özışık, 1989, 1993)

$$\beta_m L \tan \beta_m L = \omega, \tag{17}$$

which is a transcendental equation when $\omega = HL > 0$ is known. Numerical values are given in Table 4.20 of Abramowitz and Stegun (1964) and in Appendix II of Özışık (1993). This equation was solved by Burniston and Siewert (1973) where it was transformed with the substitution $\beta_m L = i\omega z$ to one where the roots are to be determined for the equation

$$z - \frac{1}{2\omega}\left[\ln\left(\frac{z-1}{z+1}\right) \pm 2m\pi i\right] = 0, \tag{18}$$

where the symbol "ln" denotes the principal branch of the natural logarithm function in the plane cut from -1 to 1 along the real axis. For

$$\Lambda_0(z) = z\left[z - \frac{1}{2\omega}\ln\left(\frac{z-1}{z+1}\right)\right], \tag{19a}$$

$$z\Lambda_m(z) = \Lambda_0(z) - im\pi z/\omega, \tag{19b}$$

the results for the positive roots are (Burniston and Siewert, 1973)

$$\beta_0 L = \left(\frac{\pi\omega}{2}\right)^{1/2} \exp\left\{-\frac{1}{\pi}\int_0^1 \left[\arg \Lambda_0^+(\nu) + \frac{\pi}{2}\right]\frac{d\nu}{\nu}\right\}, \tag{20a}$$

and

$$\beta_m L = \frac{\pi}{2}(4m^2 - 1)^{1/2} \exp\left[-\frac{1}{\pi}\int_0^1 \arg \Omega_m^+(\nu)\frac{d\nu}{\nu}\right], \quad m = 1, 2, \ldots, \tag{20b}$$

where

$$\Lambda_0^+(\nu) = \nu\left[\nu - \frac{1}{2\omega}\ln\left(\frac{1-\nu}{1+\nu}\right)\right] - \pi i\nu/2\omega, \tag{21a}$$

$$\nu^2\Omega_m^+(\nu) = [\Lambda_0^+(\nu)]^2 + m^2\pi^2\nu^2/\omega^2. \tag{21b}$$

Fixed Surface Temperature. We next consider the case where one surface is at temperature T_B and shift the temperature variable so that the surface temperature is 0. Then from Eqs. (15)–(17) we set $\alpha_1 = 1$, $H_1 = H$, $\alpha_2 = 0$ or $H_2 = 1$ (or, alternatively, $\alpha_1 = 0$, $H_1 = 1$, $\alpha_2 = 1$ or $H_2 = H$). The equation for the eigenvalues $\pm\beta_m$ is (Özışık, 1989, 1993)

$$\beta_m L \cot \beta_m L = -HL. \tag{22}$$

Numerical values are given in Table 4.19 of Abramowitz and Stegun (1964) and in Appendix II of Özışık (1993). Burniston and Siewert (1973) showed that the roots of Eq. (22) are

$$\beta_m L = m\pi \exp\left[\frac{1}{\pi}\int_0^1 \arg \Omega_m^+(\nu)\frac{d\nu}{\nu}\right], \quad m = 1, 2, \ldots, \tag{23}$$

where, for known values of $\omega = -1/HL < 0$,

$$\Omega_m^+(\nu) = [\Lambda_0^+(\nu)]^2 + m^2\pi^2\omega^2\nu^2, \tag{24a}$$

and

$$\Lambda_0^+(\nu) = 1 + \frac{\omega\nu}{2}\ln\left(\frac{1-\nu}{1+\nu}\right) + \pi i\omega\nu/2. \tag{24b}$$

4 Radiative Transfer Problems

The Wien Displacement Law. The equation for the blackbody emissive power $E_{b\lambda}$ as a function of wavelength λ and absolute temperature T is

$$E_{b\lambda} = \frac{C_1\lambda^{-5}}{\exp(C_2/\lambda T) - 1}, \tag{25}$$

where $C_1 = 3.742 \times 10^8$ W μm^4/m^2 and $C_2 = 1.4389 \times 10^4$ μm K. From $dE_{b\lambda}/d\lambda = 0$ it follows that the wavelength λ_m for the maximum power is the solution of the transcendental equation

$$(5 - x)\exp(x) = 5, \tag{26}$$

where $x = C_2/\lambda_m T$. The solution has been given as (Siewert, 1981)

$$x = 4\exp\left\{-\frac{1}{\pi}\int_0^\infty\left[\tan^{-1}\left(\frac{\pi}{\ln 5 - 5 - t - \ln t}\right) - \pi\right] \times \frac{dt}{t+5}\right\}, \tag{27}$$

which gives a value of $x = 4.96511\ldots$.

The Eigenvalues of the Radiative Transfer Equation. Siewert (1980) developed closed-form solutions for computing the largest three eigenvalues ν_j, $j = 0$, 1, and 2, for the radiative transfer equation

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu) + I(\tau,\mu) = \frac{\eta}{2}\sum_{l=0}^{L}(2l+1)f_l P_l(\mu)\int_{-1}^{1}P_l(\mu')I(\tau,\mu')d\mu', \tag{28}$$

that arise after the substitution

$$I_\nu(\tau,\mu) = \phi(\nu,\mu)\exp(-\tau/\nu). \tag{29}$$

(The eigenvalues ν_1 and ν_2 can occur when the scattering is more than linearly anisotropic.) The eigenvalues can be computed as roots of the equation

$$\Lambda(z) = 1 - \frac{\eta z}{2}\int_{-1}^{1}\frac{g(\mu,\mu)d\mu}{z-\mu}, \tag{30}$$

where

$$g(z,z) = \sum_{l=0}^{L}(2l+1)f_l g_l(z)P_l(z) \tag{31}$$

and $g_l(z)$ are the Chandrasekhar (1960) polynomials

$$(l+1)g_{l+1}(z) - zh_l g_l(z) + lg_{l-1}(z) = 0 \tag{32}$$

with $h_l = (2l+1)(1 - \eta f_l)$ and $g_0(z) = 1$ and $g_{-1}(z) = 0$. Note Eq. (7) is a special case of Eq. (30) for the case of isotropic scattering with $g(\mu, \mu) = 1$. The generalizations of Eqs. (13) for the closed-form equations for ν_0^2, the square of the largest eigenvalue, are

$$\nu_0^2 = \frac{1}{\Lambda(\infty)}\exp\left[-\frac{2}{\pi}\int_0^1\tan^{-1}\left(\frac{\pi\eta\nu g(\nu,\nu)/2}{\lambda(\nu)}\right)\frac{d\nu}{\nu}\right], \tag{33a}$$

$$\nu_0^2 = 1 - \frac{2}{\pi}\int_0^1\tan^{-1}\left(\frac{\pi\eta\nu g(\nu,\nu)/2}{\lambda(\nu)}\right)\nu d\nu + \frac{\eta}{\Lambda(\infty)}\sum_{l=0}^{L}f_l B_l \tag{33b}$$

and the generalizations of the $\Lambda(\infty)$ and $\lambda(\nu)$ in Eq. (9) are given by

$$\Lambda(\infty) = \prod_{l=0}^{L}(1 - \eta f_l), \tag{34a}$$

$$\lambda(\nu) = 1 + \frac{\eta\nu}{2}g(\nu,\nu)\ln\left(\frac{1-\nu}{1+\nu}\right) + \eta\nu\sum_{l=1}^{L}(2l+1)f_l g_l(\nu)\Gamma_l(\nu). \tag{34b}$$

Siewert (1980) has given equations for computing the B_l and $\Gamma_l(\nu)$.

5 Numerical Tests

All the explicit solutions were numerically verified by comparing their values to those obtained by iterative solution of the corresponding implicit equations for a variety of values of the independent variables. Roots of the implicit equations were found to a precision of 10^{-15} using a combination of the Newton–Raphson and bisection methods.

Integrals for the explicit solutions other than the radiative transfer eigenvalues were evaluated using Simpson's rule and were evaluated with sufficient precision to obtain roots that agreed to within 10^{-6} of the implicit solutions. For the thin fin problem, the integral was divided into two integrals of the ranges $0 \le \nu \le 0.95$ and $0.95 \le \nu \le 1.0$, and equally spaced points along the abscissa were concentrated in the second integral. Agreement of the roots of the implicit solutions over the full range of the parameter η required between 10^3 and 7×10^4 points, depending on the value of η. For the conduction problems, agreement in the solutions was verified over a large range of the parameter and for different eigenvalue indices m. Agreement in the roots for the fixed surface temperature problem for $m = 1$ and $-5 \le \omega \le 0$ required 6.5×10^4 integration points. Agreement in the roots for the insulated surface problem for $m = 1$ and $0 \le \omega \le 10$ required between 3.2×10^4 and 1.3×10^5 points, depending on the value of ω. Finally, verification of the solution to the Wien equation required 1.6×10^4 points.

For the radiative transfer eigenvalue problem, integration in the explicit solution was performed using the Clenshaw–Curtis quadrature procedure of MAPLE V (Char et al., 1992). Agreement in the eigenvalue was obtained to at least nine significant digits using both Eq. (33*a*) and Eq. (33*b*) for a fourth-order binomial model of the coefficients f_l. It was observed it was much easier to obtain good numerical accuracy with Eq. (33*b*) than with Eq. (33*a*).

6 Summary

We have shown there is a rich literature of closed-form solutions of transcendental equations that can be taken over to five classical conduction, convection, and radiation heat transfer problems. All solutions evolve from the same canonical form given by Eq. (4) but are so complicated they reveal little physical insight. While it can hardly be recommended that these complicated integrals are easier to implement than a straightforward iteration routine, the solutions of the associated transcendental equations contribute an interesting perspective to the folklore surrounding the solution of such problems.

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